

Explicit Determination of the Picard Group of Moduli Spaces of Semistable G -Bundles on Curves

Arzu Boysal and Shrawan Kumar

Introduction

Let G be a connected, simply-connected, simple affine algebraic group and \mathcal{C}_g be a smooth irreducible projective curve of any genus $g \geq 1$ over \mathbb{C} . Denote by $\mathfrak{M}_{\mathcal{C}_g}(G)$ the moduli space of semistable principal G -bundles on \mathcal{C}_g . Let $\text{Pic}(\mathfrak{M}_{\mathcal{C}_g}(G))$ be the Picard group of $\mathfrak{M}_{\mathcal{C}_g}(G)$ and let X be the infinite Grassmannian of the affine Kac-Moody group associated to G . It is known that $\text{Pic}(X) \simeq \mathbb{Z}$ and is generated by a homogenous line bundle \mathcal{L}_{χ_0} . Also, as proved by Kumar-Narasimhan [KN], there exists a canonical injective group homomorphism

$$\beta : \text{Pic}(\mathfrak{M}_{\mathcal{C}_g}(G)) \hookrightarrow \text{Pic}(X),$$

which takes $\Theta_V(\mathcal{C}_g, G) \mapsto \mathcal{L}_{\chi_0}^{m_V}$ for any finite dimensional representation V of G , where $\Theta_V(\mathcal{C}_g, G)$ is the theta bundle associated to the G -module V and m_V is its Dynkin index (cf. Theorem 2.2). As an immediate corollary, they obtained that

$$\text{Pic}(\mathfrak{M}_{\mathcal{C}_g}(G)) \simeq \mathbb{Z},$$

generalizing the corresponding result for $G = SL(n)$ proved by Drezet-Narasimhan [DN]. However, the precise image of β was not known for non-classical G excluding G_2 . (For classical G and G_2 , see [KN], [LS], [BLS].) The main aim of this paper is to determine the image of β for an arbitrary G . It is shown that the image of β is generated by $\mathcal{L}_{\chi_0}^{m_G}$, where m_G is the least common multiple of the coefficients of the coroot θ^\vee written in terms of the simple coroots, θ being the highest root of G (cf. Theorem 2.4, see also Proposition 2.3 where m_G is explicitly given for each G). As a consequence, we obtain that the theta bundles $\Theta_V(\mathcal{C}_g, G)$, where V runs over all the finite dimensional representations of G , generate $\text{Pic}(\mathfrak{M}_{\mathcal{C}_g}(G))$ (cf. Theorem 1.3). In fact, it is shown that there is a fundamental weight ω_d such that the theta bundle $\Theta_{V(\omega_d)}(\mathcal{C}_g, G)$ corresponding to the irreducible highest

weight G -module $V(\omega_d)$ with highest weight ω_d generates $\text{Pic}(\mathfrak{M}_{\mathcal{C}_g}(G))$ (cf. Theorem 2.4). All these fundamental weights ω_d are explicitly determined in Proposition 2.3.

It may be mentioned that Picard group of the moduli *stack* of G -bundles is studied in [LS], [BLS], [T₂].

We now briefly outline the idea of the proofs. Recall that, by a celebrated result of Narasimhan-Seshadri, the underlying real analytic space $M_g(G)$ of $\mathfrak{M}_{\mathcal{C}_g}(G)$ admits a description as the space of representations of the fundamental group $\pi_1(\mathcal{C}_g)$ into a fixed compact form of G up to conjugation. In particular, $M_g(G)$ depends only upon g and G (and not on the specific choice of the projective curve \mathcal{C}_g). Moreover, this description gives rise to a standard embedding $i_g : M_g(G) \hookrightarrow M_{g+1}(G)$.

Let V be any finite dimensional representation of G . We first show that the first Chern class of the theta bundle $\Theta_V(\mathcal{C}_g, G)$ does not depend upon the choice of the smooth projective curve \mathcal{C}_g , as long as g is fixed (cf. Proposition 1.6).

We next show that the first Chern class of $\Theta_V(\mathcal{C}_{g+1}, G)$ restricts to the first Chern class of $\Theta_V(\mathcal{C}_g, G)$ under the embedding i_g (cf. Proposition 1.8). This result is proved by first reducing the case of general G to $SL(n)$ and then reducing the case of $SL(n)$ to $SL(2)$. The corresponding result for $SL(2)$ is obtained by showing that the inclusion $M_g(SL(2)) \hookrightarrow M_{g+1}(SL(2))$ induces isomorphism in cohomology $H^2(M_{g+1}(SL(2)), \mathbb{Z}) \simeq H^2(M_g(SL(2)), \mathbb{Z})$ (cf. Proposition 1.7). The last result for H^2 with rational coefficients is fairly well known (and follows easily by observing that the symplectic form on $M_{g+1}(G)$ restricts to the symplectic form on $M_g(G)$) but the result with integral coefficients is more delicate and is proved in Section 4. The proof involves the calculation of the determinant bundle of the Poincaré bundle on $\mathcal{C}_g \times \mathcal{J}_{\mathcal{C}_g}$, $\mathcal{J}_{\mathcal{C}_g}$ being the Jacobian of \mathcal{C}_g which consists of the isomorphism classes of degree 0 line bundles on \mathcal{C}_g .

By virtue of the above mentioned two propositions (Propositions 1.6 and 1.8), to prove our main result determining $\text{Pic}(\mathfrak{M}_{\mathcal{C}_g}(G))$ stated in the first paragraph for any $g \geq 1$, it suffices to consider the case of genus $g = 1$.

In the genus $g = 1$ case, $\mathfrak{M}_{\mathcal{C}_1}(G)$ admits a description as the weighted projective space $\mathbb{P}(1, a_1^\vee, a_2^\vee, \dots, a_k^\vee)$, where a_i^\vee are the coefficients of the coroot θ^\vee written in terms of the simple coroots and k is the rank of G (cf. Theorems 3.1 and 3.3). The ample generator of the Picard group of $\mathbb{P}(1, a_1^\vee, a_2^\vee, \dots, a_k^\vee)$ is known to be $\mathcal{O}_{\mathbb{P}(1, a_1^\vee, a_2^\vee, \dots, a_k^\vee)}(m_G)$ (cf. Theorem 3.4). In section 3, we show that $\Theta_{V(\omega_d)}(\mathcal{C}_1, G)$ is, in fact, $\mathcal{O}_{\mathbb{P}(1, a_1^\vee, a_2^\vee, \dots, a_k^\vee)}(m_G)$, and hence it is the ample generator of $\text{Pic}(\mathfrak{M}_{\mathcal{C}_1}(G))$. The proof makes use of the Verlinde formula determining the dimension of the space of global

sections $H^0(\mathfrak{M}_{\mathcal{C}_g}(G), \mathcal{L})$ (cf. Theorem 3.5).

We thank P. Belkale, L. Jeffrey and M.S. Narasimhan for some helpful correspondences/conversations. We are grateful to the referee for some very helpful comments; in particular, for pointing out Theorem 3.3, which simplified our original proof of Proposition 1.9. The second author was partially supported by the NSF.

1 Statement of the Main Theorem and its Proof

For a topological space X , $H^i(X)$ denotes the singular cohomology of X with integral coefficients, unless otherwise explicitly stated.

Let G be a connected, simply-connected, simple affine algebraic group over \mathbb{C} . This will be our tacit assumption on G throughout the paper. Let \mathcal{C}_g be a smooth irreducible projective curve (over \mathbb{C}) of genus g , which we assume to be ≥ 1 . Let $\mathfrak{M}_{\mathcal{C}_g} = \mathfrak{M}_{\mathcal{C}_g}(G)$ be the moduli space of semistable principal G -bundles on \mathcal{C}_g .

We begin by recalling the following result due to Kumar-Narasimhan [KN, Theorem 2.4]. (In loc. cit. the genus g is assumed to be ≥ 2 . For the genus $g = 1$ case, the result follows from Theorems 3.1, 3.3 and 3.4.)

1.1 Theorem. *With the notation as above,*

$$\mathrm{Pic}(\mathfrak{M}_{\mathcal{C}_g}) \simeq \mathbb{Z},$$

where $\mathrm{Pic}(\mathfrak{M}_{\mathcal{C}_g})$ is the group of isomorphism classes of algebraic line bundles on $\mathfrak{M}_{\mathcal{C}_g}$.

In particular, any nontrivial line bundle on $\mathfrak{M}_{\mathcal{C}_g}$ is ample or its inverse is ample.

1.2 Definition. Let \mathcal{F} be a family of vector bundles on \mathcal{C}_g parametrized by a variety X , i.e., \mathcal{F} is a vector bundle over $\mathcal{C}_g \times X$. Then, the ‘determinant of the cohomology’ gives rise to the determinant bundle $\mathrm{Det}(\mathcal{F})$ of the family \mathcal{F} , which is a line bundle over the base X . By definition, the fiber of $\mathrm{Det}(\mathcal{F})$ over any $x \in X$ is given by the expression:

$$\mathrm{Det}(\mathcal{F})|_x = \wedge^{\mathrm{top}}(H^0(\mathcal{C}_g, \mathcal{F}_x))^* \otimes \wedge^{\mathrm{top}}(H^1(\mathcal{C}_g, \mathcal{F}_x)),$$

where \mathcal{F}_x is the restriction of \mathcal{F} to $\mathcal{C}_g \times x$ (cf., e.g., [L, Chap. 6, §1], [KM]).

Let $\mathcal{R}(G)$ denote the set of isomorphism classes of all the finite dimensional algebraic representations of G . For any V in $\mathcal{R}(G)$, we have the

Θ -bundle $\Theta_V(\mathcal{C}_g) = \Theta_V(\mathcal{C}_g, G)$ on $\mathfrak{M}_{\mathcal{C}_g}$, which is an algebraic line bundle whose fibre at any principal G -bundle $E \in \mathfrak{M}_{\mathcal{C}_g}$ is given by the expression

$$\Theta_V(\mathcal{C}_g)|_E = \wedge^{\text{top}}(H^0(\mathcal{C}_g, E_V))^* \otimes \wedge^{\text{top}}(H^1(\mathcal{C}_g, E_V)),$$

where E_V is the associated vector bundle $E \times_G V$ on \mathcal{C}_g . Observe that the moduli space $\mathfrak{M}_{\mathcal{C}_g}$ does not parametrize a universal family of G -bundles, however, the theta bundle $\Theta_V(\mathcal{C}_g)$ (which is essentially the determinant bundle if there were a universal family parametrized by $\mathfrak{M}_{\mathcal{C}_g}$) still exists (cf. [K₁, §3.7]).

Now, we can state the main result of this paper.

1.3 Theorem.

$$\text{Pic}(\mathfrak{M}_{\mathcal{C}_g}) = \langle \Theta_V(\mathcal{C}_g), V \in \mathcal{R}(G) \rangle,$$

where the notation $\langle \rangle$ denotes the group generated by the elements in the bracket.

1.4 Lemma.

$$c : \text{Pic}(\mathfrak{M}_{\mathcal{C}_g}) \simeq H^2(\mathfrak{M}_{\mathcal{C}_g}, \mathbb{Z}),$$

where c maps any line bundle \mathfrak{L} to its first Chern class $c_1(\mathfrak{L})$.

In particular,

$$H^2(\mathfrak{M}_{\mathcal{C}_g}, \mathbb{Z}) \simeq \mathbb{Z}.$$

The first Chern class of the ample generator of $\text{Pic}(\mathfrak{M}_{\mathcal{C}_g})$ is called the positive generator of $H^2(\mathfrak{M}_{\mathcal{C}_g}, \mathbb{Z})$.

Proof. Consider the following exact sequence of abelian groups:

$$0 \rightarrow \mathbb{Z} \rightarrow \mathbb{C} \xrightarrow{f} \mathbb{C}^* \rightarrow 0,$$

where $f(x) = e^{2\pi i x}$. This gives rise to the following exact sequence of sheaves on $\mathfrak{M}_{\mathcal{C}_g}$ endowed with the analytic topology:

$$0 \rightarrow \bar{\mathbb{Z}} \rightarrow \bar{\mathcal{O}}_{\mathfrak{M}_{\mathcal{C}_g}} \rightarrow \bar{\mathcal{O}}_{\mathfrak{M}_{\mathcal{C}_g}}^* \rightarrow 0,$$

where $\bar{\mathcal{O}}_{\mathfrak{M}_{\mathcal{C}_g}}$ is the sheaf of holomorphic functions on $\mathfrak{M}_{\mathcal{C}_g}$, $\bar{\mathcal{O}}_{\mathfrak{M}_{\mathcal{C}_g}}^*$ is the sheaf of invertible elements of $\bar{\mathcal{O}}_{\mathfrak{M}_{\mathcal{C}_g}}$ and $\bar{\mathbb{Z}}$ is the constant sheaf corresponding to the abelian group \mathbb{Z} .

The above sequence, of course, induces the following long exact sequence in cohomology:

$$\cdots \rightarrow H^1(\mathfrak{M}_{\mathcal{C}_g}, \bar{\mathcal{O}}_{\mathfrak{M}_{\mathcal{C}_g}}) \rightarrow H^1(\mathfrak{M}_{\mathcal{C}_g}, \bar{\mathcal{O}}_{\mathfrak{M}_{\mathcal{C}_g}}^*) \xrightarrow{\bar{c}} H^2(\mathfrak{M}_{\mathcal{C}_g}, \mathbb{Z}) \rightarrow H^2(\mathfrak{M}_{\mathcal{C}_g}, \bar{\mathcal{O}}_{\mathfrak{M}_{\mathcal{C}_g}}) \rightarrow \cdots.$$

First of all,

$$(1) \quad \text{Pic}(\mathfrak{M}_{\mathcal{C}_g}) \simeq H^1(\mathfrak{M}_{\mathcal{C}_g}, \mathcal{O}_{\mathfrak{M}_{\mathcal{C}_g}}^*),$$

where $\mathcal{O}_{\mathfrak{M}_{\mathcal{C}_g}}$ is the sheaf of algebraic functions on $\mathfrak{M}_{\mathcal{C}_g}$ and $\mathcal{O}_{\mathfrak{M}_{\mathcal{C}_g}}^*$ is the subsheaf of invertible elements of $\mathcal{O}_{\mathfrak{M}_{\mathcal{C}_g}}$.

Moreover, by GAGA, $\mathfrak{M}_{\mathcal{C}_g}$ being a projective variety,

$$(2) \quad H^1(\mathfrak{M}_{\mathcal{C}_g}, \mathcal{O}_{\mathfrak{M}_{\mathcal{C}_g}}^*) \simeq H^1(\mathfrak{M}_{\mathcal{C}_g}, \bar{\mathcal{O}}_{\mathfrak{M}_{\mathcal{C}_g}}^*),$$

and also, for any $p \geq 0$,

$$(3) \quad H^p(\mathfrak{M}_{\mathcal{C}_g}, \mathcal{O}_{\mathfrak{M}_{\mathcal{C}_g}}) \simeq H^p(\mathfrak{M}_{\mathcal{C}_g}, \bar{\mathcal{O}}_{\mathfrak{M}_{\mathcal{C}_g}}).$$

By Kumar-Narasimhan [KN, Theorem 2.8], $H^i(\mathfrak{M}_{\mathcal{C}_g}, \mathcal{O}_{\mathfrak{M}_{\mathcal{C}_g}}) = 0$ for $i > 0$. Hence, under the identification (1), by (2)-(3) and the above long exact cohomology sequence,

$$\text{Pic}(\mathfrak{M}_{\mathcal{C}_g}) \xrightarrow{\simeq^c} H^2(\mathfrak{M}_{\mathcal{C}_g}, \mathbb{Z}),$$

where c is the map \bar{c} under the above identifications. Moreover, as is well known, c is the first Chern class map. \square

Let us fix a maximal compact subgroup K of G . Denote the Riemann surface with g handles, considered only as a topological manifold, by C_g . Thus, the underlying topological manifold of \mathcal{C}_g is C_g . Define $M_g(G) := \varphi^{-1}(1)/\text{Ad}K$, where $\varphi : K^{2g} \rightarrow K$ is the commutator map $\varphi(k_1, k_2, \dots, k_{2g}) = [k_1, k_2][k_3, k_4] \cdots [k_{2g-1}, k_{2g}]$ and $\varphi^{-1}(1)/\text{Ad}K$ refers to the quotient of $\varphi^{-1}(1)$ by K under the diagonal adjoint action of K on K^{2g} .

Now, we recall the following fundamental result due to Narasimhan-Seshadri [NS] for vector bundles and extended for an arbitrary G by Ramanathan [R₁, R₂].

Consider the standard generators $a_1, b_1, a_2, b_2, \dots, a_g, b_g$ of $\pi_1(\mathcal{C}_g)$ (cf. [N, §14]). Then, we have the presentation:

$$\pi_1(\mathcal{C}_g) = F[a_1, \dots, a_g, b_1, \dots, b_g] / \langle [a_1, b_1] \cdots [a_g, b_g] \rangle,$$

where $F[a_1, \dots, a_g, b_1, \dots, b_g]$ denotes the free group generated by $a_1, \dots, a_g, b_1, \dots, b_g$ and $\langle [a_1, b_1] \cdots [a_g, b_g] \rangle$ denotes the normal subgroup generated by the single element $[a_1, b_1] \cdots [a_g, b_g]$.

1.5 Theorem. *Having chosen the standard generators $a_1, b_1, a_2, b_2, \dots, a_g, b_g$ of $\pi_1(\mathcal{C}_g)$, there exists a canonical isomorphism of real analytic spaces:*

$$\theta_{\mathcal{C}_g}(G) : M_g(G) \simeq \mathfrak{M}_{\mathcal{C}_g}(G).$$

In the sequel, we will often make this identification.

1.6 Proposition. *For any $V \in \mathcal{R}(G)$, $c(\Theta_V(\mathcal{C}_g, G))$, under the above identification $\theta_{\mathcal{C}_g}(G)$, does not depend on the choice of the projective variety structure \mathcal{C}_g on the Riemann surface C_g for any fixed g .*

Proof. Let $\rho : G \rightarrow SL(V)$ be the given representation. By taking a K -invariant Hermitian form on V we get $\rho(K) \subset SU(n)$, where $n = \dim V$. For any principal G -bundle E on \mathcal{C}_g , let $E_{SL(V)}$ be the principal $SL(V)$ -bundle over \mathcal{C}_g obtained by the extension of the structure group via ρ . Then, if E is semistable, so is $E_{SL(V)}$, giving rise to a variety morphism $\hat{\rho} : \mathfrak{M}_{\mathcal{C}_g}(G) \rightarrow \mathfrak{M}_{\mathcal{C}_g}(SL(V))$ (cf. [RR, Theorem 3.18]). Hence, we get the commutative diagram:

$$(D_1) \quad \begin{array}{ccc} \mathfrak{M}_{\mathcal{C}_g}(G) & \xrightarrow{\hat{\rho}} & \mathfrak{M}_{\mathcal{C}_g}(SL(V)) \\ \uparrow & & \uparrow \\ M_g(G) & \xrightarrow{\bar{\rho}} & M_g(SL(V)), \end{array}$$

where $\bar{\rho}$ is induced from the commutative diagram:

$$\begin{array}{ccc} K^{2g} & \xrightarrow{\varphi} & K \\ \downarrow \rho^{\times 2g} & & \downarrow \rho \\ SU(n)^{2g} & \xrightarrow{\varphi} & SU(n). \end{array}$$

The diagram (D_1) induces the following commutative diagram in cohomology:

$$(D_2) \quad \begin{array}{ccc} H^2(\mathfrak{M}_{\mathcal{C}_g}(SL(V)), \mathbb{Z}) & \xrightarrow{\hat{\rho}^*} & H^2(\mathfrak{M}_{\mathcal{C}_g}(G), \mathbb{Z}) \\ \downarrow \parallel & & \downarrow \parallel \\ H^2(M_g(SL(V)), \mathbb{Z}) & \xrightarrow{\bar{\rho}^*} & H^2(M_g(G), \mathbb{Z}). \end{array}$$

By the construction of the Θ -bundle, $\hat{\rho}^*(\Theta_V(\mathcal{C}_g, SL(V))) = \Theta_V(\mathcal{C}_g, G)$, where $\hat{\rho}^*$ also denotes the pullback of line bundles and V is thought of as the standard representation of $SL(V)$.

Thus, using the functoriality of the Chern class, we get

$$(1) \quad \hat{\rho}^*(c(\Theta_V(\mathcal{C}_g, SL(V)))) = c(\Theta_V(\mathcal{C}_g, G)).$$

By Drezet-Narasimhan [DN], $c(\Theta_V(\mathcal{C}_g, SL(V)))$ is the unique positive generator of $H^2(\mathfrak{M}_{\mathcal{C}_g}(SL(V)), \mathbb{Z})$ and thus is independent of the choice of \mathcal{C}_g under the identification $\theta_{\mathcal{C}_g}(SL(V))^*$. Consequently, by (1) and the above commutative diagram (D₂), $c(\Theta_V(\mathcal{C}_g, G))$ is independent of the choice of \mathcal{C}_g . \square

From now on we will denote the cohomology class $c(\Theta_V(\mathcal{C}_g, G))$ in $H^2(M_g(G), \mathbb{Z})$, under the identification $\theta_{\mathcal{C}_g}(G)^*$, by $c(\Theta_V(g, G))$.

Consider the embedding

$$i_g = i_g(G) : M_g(G) \hookrightarrow M_{g+1}(G)$$

induced by the inclusion of $K^{2g} \rightarrow K^{2g+2}$ via $(k_1, \dots, k_{2g}) \mapsto (k_1, \dots, k_{2g}, 1, 1)$.

By virtue of the map i_g , we will identify $M_g(G)$ as a subspace of $M_{g+1}(G)$. In particular, we get the following induced sequence of maps in the second cohomology.

$$H^2(M_1(G), \mathbb{Z}) \xleftarrow{i_1^*} H^2(M_2(G), \mathbb{Z}) \xleftarrow{i_2^*} H^2(M_3(G), \mathbb{Z}) \xleftarrow{i_3^*} \dots$$

1.7 Proposition. *For $G = SL(2)$, the maps $i_g^* : H^2(M_{g+1}(G), \mathbb{Z}) \rightarrow H^2(M_g(G), \mathbb{Z})$ are isomorphisms for any $g \geq 1$.*

In particular, i_g^ takes the positive generator of $H^2(M_{g+1}(SL(2)), \mathbb{Z})$ to the positive generator of $H^2(M_g(SL(2)), \mathbb{Z})$.*

We shall prove this proposition in Section 4.

1.8 Proposition. *For any $V \in \mathcal{R}(G)$ and any $g \geq 1$, $i_g^*(c(\Theta_V(g+1, G))) = c(\Theta_V(g, G))$.*

Proof. We first claim that it suffices to prove the above proposition for $G = SL(n)$ and the standard n -dimensional representation V of $SL(n)$.

Let $\rho : G \rightarrow SL(V)$ be the given representation. Consider the following commutative diagram:

$$\begin{array}{ccc} M_g(G) & \xrightarrow{i_g} & M_{g+1}(G) \\ \bar{\rho} \downarrow & & \downarrow \bar{\rho} \\ M_g(SL(V)) & \xrightarrow{i_g} & M_{g+1}(SL(V)), \end{array}$$

where $\bar{\rho}$ is the map defined in the proof of Proposition 1.6. It induces the commutative diagram:

$$\begin{array}{ccc} H^2(M_g(G), \mathbb{Z}) & \xleftarrow{i_g^*} & H^2(M_{g+1}(G), \mathbb{Z}) \\ \bar{\rho}^* \uparrow & & \bar{\rho}^* \uparrow \\ H^2(M_g(SL(V)), \mathbb{Z}) & \xleftarrow{i_g^*} & H^2(M_{g+1}(SL(V)), \mathbb{Z}). \end{array}$$

Therefore, using the commutativity of the above diagram and equation (1) of Proposition 1.6, supposing that $i_g^*(c(\Theta_V(g+1, SL(V)))) = c(\Theta_V(g, SL(V)))$, we get $i_g^*(c(\Theta_V(g+1, G))) = c(\Theta_V(g, G))$. Hence, Proposition 1.8 is established for any G provided we assume its validity for $G = SL(V)$ and its standard representation in V .

We further reduce the proposition from $SL(n)$ to $SL(2)$. As in the proof of Proposition 1.6, consider the mappings

$$\bar{\rho} : M_g(SL(2)) \rightarrow M_g(SL(n)), \text{ and}$$

$$\hat{\rho} : \mathfrak{M}_{\mathcal{C}_g}(SL(2)) \rightarrow \mathfrak{M}_{\mathcal{C}_g}(SL(n))$$

induced by the inclusions

$$SU(2) \rightarrow SU(n) \text{ and } SL(2) \rightarrow SL(n),$$

given by $m \mapsto \text{diag}(m, 1, \dots, 1)$.

The maps $\bar{\rho}$ and $\hat{\rho}$ induce the commutative diagram:

$$\begin{array}{ccc} H^2(M_g(SL(n)), \mathbb{Z}) & \xrightarrow{\bar{\rho}^*} & H^2(M_g(SL(2)), \mathbb{Z}) \\ \uparrow & & \uparrow \\ H^2(\mathfrak{M}_{\mathcal{C}_g}(SL(n)), \mathbb{Z}) & \xrightarrow{\hat{\rho}^*} & H^2(\mathfrak{M}_{\mathcal{C}_g}(SL(2)), \mathbb{Z}). \end{array}$$

By the construction of the Θ -bundle, $\hat{\rho}^*(\Theta_V(\mathcal{C}_g, SL(n))) = \Theta_{V_2}(\mathcal{C}_g, SL(2))$, where V_2 is the standard 2-dimensional representation of $SL(2)$.

Thus, using the functoriality of the Chern class, we get

$$(1) \quad \hat{\rho}^*(c(\Theta_V(\mathcal{C}_g, SL(n)))) = c(\Theta_{V_2}(\mathcal{C}_g, SL(2))).$$

Using one more time the result of Drezet-Narasimhan that $c(\Theta_V(\mathcal{C}_g, SL(n)))$ is the unique positive generator of $H^2(\mathfrak{M}_{\mathcal{C}_g}(SL(n)))$ for any n (cf. Proof of Proposition 1.6), we see that $\hat{\rho}^*$ is surjective and hence an isomorphism by Lemma 1.4.

Consider the following commutative diagram:

$$\begin{array}{ccc} H^2(M_g(SL(n)), \mathbb{Z}) & \xleftarrow{i_g^*} & H^2(M_{g+1}(SL(n)), \mathbb{Z}) \\ \bar{\rho}^* \downarrow & & \bar{\rho}^* \downarrow \\ H^2(M_g(SL(2)), \mathbb{Z}) & \xleftarrow{i_g^*} & H^2(M_{g+1}(SL(2)), \mathbb{Z}). \end{array}$$

Suppose that the proposition is true for $G = SL(2)$ and the standard representation V_2 , i.e.,

$$(2) \quad i_g^*(c(\Theta_{V_2}(g+1, SL(2)))) = c(\Theta_{V_2}(g, SL(2))).$$

Then, using the commutativity of the above diagram and (1) together with the fact that $\bar{\rho}^*$ is an isomorphism, we get that $i_g^*(c(\Theta_V(g+1, SL(n)))) = c(\Theta_V(g, SL(n)))$. Finally, (2) follows from the result of Drezet-Narasimhan cited above and Proposition 1.7. Hence the proposition is established for any G (once we prove Proposition 1.7). \square

1.9 Proposition. *For $g = 1$, Theorem 1.3 is true.*

The proof of this proposition will be given in Section 3.

Proof of Theorem 1.3. Denote the subgroup $\langle \Theta_V(\mathcal{C}_g, G), V \in \mathcal{R}(G) \rangle$ of $\text{Pic}(\mathfrak{M}_{\mathcal{C}_g}(G))$ by $\text{Pic}^\Theta(\mathfrak{M}_{\mathcal{C}_g}(G))$.

Set $H_\Theta^2(M_g(G)) := c(\text{Pic}^\Theta(\mathfrak{M}_{\mathcal{C}_g}(G)))$. By virtue of Proposition 1.6, this is well defined, i.e., $H_\Theta^2(M_g(G))$ does not depend upon the choice of the projective variety structure \mathcal{C}_g on C_g . Moreover, by Proposition 1.8, $i_g^*(H_\Theta^2(M_{g+1}(G))) = H_\Theta^2(M_g(G))$.

Thus, we get the following commutative diagram, where the upward arrows are inclusions and the maps in the bottom horizontal sequence are induced from the maps i_g^* .

$$\begin{array}{ccccccc} H^2(M_1(G)) & \xleftarrow{i_1^*} & H^2(M_2(G)) & \xleftarrow{i_2^*} & H^2(M_3(G)) & \xleftarrow{i_3^*} & \dots \\ \uparrow & & \uparrow & & \uparrow & & \\ H_\Theta^2(M_1(G)) & \leftarrow & H_\Theta^2(M_2(G)) & \leftarrow & H_\Theta^2(M_3(G)) & \leftarrow & \dots \end{array}$$

By Proposition 1.9 and Lemma 1.4, $H^2(M_1(G)) = H_\Theta^2(M_1(G))$. Then, i_1^* is surjective and hence an isomorphism (by using Lemma 1.4 again). Thus, by the commutativity of the above diagram, the inclusion $H_\Theta^2(M_2(G)) \hookrightarrow H^2(M_2(G))$ is an isomorphism. Arguing the same way, we get that $H^2(M_g(G)) = H_\Theta^2(M_g(G))$ for all g . This completes the proof of the theorem by virtue of the isomorphism c of Lemma 1.4. \square

2 Comparison of the Picard Groups of $\mathfrak{M}_{\mathcal{C}_g}$ and the Infinite Grassmannian

As earlier, let G be a connected, simply-connected, simple affine algebraic group over \mathbb{C} . We fix a Borel subgroup B of G and a maximal torus $T \subset B$. Let \mathfrak{h} (resp. \mathfrak{b}) be the Lie algebra of T (resp. B). Let $\Delta^+ \subset \mathfrak{h}^*$ be the set of positive roots (i.e., the roots of \mathfrak{b} with respect to \mathfrak{h}) and let $\{\omega_i\}_{1 \leq i \leq k} \subset \mathfrak{h}^*$ be the set of fundamental weights, where k is the rank of G . As earlier, $\mathcal{R}(G)$ denotes the set of isomorphism classes of all the finite dimensional algebraic representations of G . This is a semigroup under the direct sum of two representations. Let $R(G)$ denote the associated Grothendieck group. Then, $R(G)$ is a ring, where the product is induced from the tensor product of two representations. Then, the fundamental representations $\{V(\omega_i)\}_{1 \leq i \leq k}$ generate the representation ring $R(G)$ as a ring [A].

Let X be the infinite Grassmannian associated to the affine Kac-Moody group \mathcal{G} corresponding to G , i.e., $X := \mathcal{G}/\mathcal{P}$, where \mathcal{P} is the standard maximal parabolic subgroup of \mathcal{G} (cf. [K₂, §13.2.12]; in loc. cit., X is denoted by $\mathcal{Y} = \mathcal{X}^Y$). It is known that $\text{Pic}(X)$ is isomorphic to \mathbb{Z} and is generated by the homogenous line bundle \mathcal{L}_{χ_0} (cf. [K₂, Proposition 13.2.19]).

We recall the following definition from [D, §2].

2.1 Definition. Let \mathfrak{g}_1 and \mathfrak{g}_2 be two (finite dimensional) complex simple Lie algebras and $\varphi : \mathfrak{g}_1 \rightarrow \mathfrak{g}_2$ be a Lie algebra homomorphism. There exists a unique number $m_\varphi \in \mathbb{C}$, called the *Dynkin index* of the homomorphism φ , satisfying

$$\langle \varphi(x), \varphi(y) \rangle = m_\varphi \langle x, y \rangle, \text{ for all } x, y \in \mathfrak{g}_1,$$

where \langle, \rangle is the Killing form on \mathfrak{g}_1 (and \mathfrak{g}_2) normalized so that $\langle \theta, \theta \rangle = 2$ for the highest root θ .

For a Lie algebra \mathfrak{g}_1 as above and a finite dimensional representation V of \mathfrak{g}_1 , by the *Dynkin index* m_V of V , we mean the Dynkin index of the Lie algebra homomorphism $\rho : \mathfrak{g}_1 \rightarrow \mathfrak{sl}(V)$, where $\mathfrak{sl}(V)$ is the Lie algebra of trace 0 endomorphisms of V .

Then, for any two finite dimensional representations V and W of \mathfrak{g}_1 , we have, by [D, Chap. 1, §2] or [KN, Lemma 4.5],

$$(1) \quad m_{V \otimes W} = m_V \dim W + m_W \dim V.$$

We recall the following main result of Kumar-Narasimhan [KN, Theorem 2.4].

2.2 Theorem. *There exists a ‘natural’ injective group homomorphism*

$$\beta : \text{Pic}(\mathfrak{M}_{\mathcal{C}_g}(G)) \hookrightarrow \text{Pic}(X).$$

Moreover, by [KNR, Theorem 5.4] (see also [Fa]), for any $V \in \mathcal{R}(G)$,

$$(1) \quad \beta(\Theta_V(\mathcal{C}_g, G)) = \mathfrak{L}_{\chi_0}^{\otimes m_V},$$

where V is thought of as a module for \mathfrak{g} under differentiation and m_V is its Dynkin index.

We also recall the following result from [D, Table 5], [KN, Proposition 4.7], or [LS, §2].

2.3 Proposition. *For any simple Lie algebra \mathfrak{g} , there exists a (not unique in general) fundamental weight ω_d such that $m_{V(\omega_d)}$ divides each of $\{m_{V(\omega_i)}\}_{1 \leq i \leq k}$. Thus, by (1) of Definition 2.1, $m_{V(\omega_d)}$ divides m_V for any $V \in \mathcal{R}(G)$.*

The following table gives the list of all such ω_d ’s and the corresponding Dynkin index $m_{V(\omega_d)}$.

Type of G	ω_d	$m_{V(\omega_d)}$
A_k ($k \geq 1$)	ω_1, ω_k	1
C_k ($k \geq 2$)	ω_1	1
B_k ($k \geq 3$)	ω_1	2
D_k ($k \geq 4$)	ω_1	2
G_2	ω_1	2
F_4	ω_4	6
E_6	ω_1, ω_6	6
E_7	ω_7	12
E_8	ω_8	60.

For B_3 , ω_3 also satisfies $m_{V(\omega_3)} = 2$; for D_4 , ω_3 and ω_4 both have $m_{V(\omega_3)} = m_{V(\omega_4)} = 2$.

Let θ be the highest root of G . Observe that, for any G , $m_{V(\omega_d)}$ is the least common multiple of the coefficients of the coroot θ^\vee written in terms of the simple coroots. We shall denote $m_{V(\omega_d)}$ by m_G .

Combining the above result with Theorem 1.3, we get the following.

2.4 Theorem. *For any \mathcal{C}_g with $g \geq 1$ and G as in Section 1, the Picard group $\text{Pic}(\mathfrak{M}_{\mathcal{C}_g}(G))$ is freely generated by the Θ -bundle $\Theta_{V(\omega_d)}(\mathcal{C}_g, G)$, where ω_d is any fundamental weight as in the above proposition.*

In particular,

$$(1) \quad \text{Im}(\beta) \text{ is freely generated by } \mathfrak{L}_{\chi_0}^{\otimes m_G}.$$

Proof. By Theorem 1.3,

$$\mathrm{Pic}(\mathfrak{M}_{\mathcal{C}_g}(G)) = \langle \Theta_V(\mathcal{C}_g, G), V \in \mathcal{R}(G) \rangle .$$

Thus, by Theorem 2.2 and Proposition 2.3,

$$\mathrm{Im}(\beta) = \langle \mathfrak{L}_{\chi_0}^{\otimes m_V}, V \in \mathcal{R}(G) \rangle = \langle \mathfrak{L}_{\chi_0}^{\otimes m_G} \rangle .$$

This proves (1).

Since β is injective, by the above description of $\mathrm{Im}(\beta)$, $\Theta_{V(\omega_d)}(\mathcal{C}_g, G)$ freely generates $\mathrm{Pic}(\mathfrak{M}_{\mathcal{C}_g}(G))$, proving the theorem. \square

Following the same argument as in [So, §4], using Theorem 2.4 and Proposition 2.3, we get the following corollary for genus $g \geq 2$. For genus $g = 1$, use Theorems 3.1 and 3.3 together with [BR, Theorem 7.1.d]. This corollary is due to [BLS], [So].

2.5 Corollary. *Let G be any group and \mathcal{C}_g be any curve as in Section 1. Then, the moduli space $\mathfrak{M}_{\mathcal{C}_g}(G)$ is locally factorial if and only if G is of type A_k ($k \geq 1$) or C_k ($k \geq 2$).*

3 Proof of Proposition 1.9

Let G be as in the beginning of Section 1. In this section, we identify $\mathfrak{M}_{\mathcal{C}_1}(G)$ with a weighted projective space and show that the generator of $\mathrm{Pic}(\mathfrak{M}_{\mathcal{C}_1}(G))$ is $\Theta_{V(\omega_d)}(\mathcal{C}_1, G)$ as claimed.

We recall the following theorem due independently to Laszlo [La, Theorem 4.16] and Friedman-Morgan-Witten [FMW, §2].

3.1 Theorem. *Let \mathcal{C}_1 be a smooth, irreducible projective curve of genus 1. Then, there is a natural variety isomorphism between the moduli space $\mathfrak{M}_{\mathcal{C}_1}(G)$ and $(\mathcal{C}_1 \otimes_{\mathbb{Z}} Q^{\vee})/W$, where Q^{\vee} is the coroot lattice of G and W is its Weyl group acting canonically on Q^{\vee} (and acting trivially on \mathcal{C}_1).*

3.2 Definition. Let $N = (n_0, \dots, n_k)$ be a $k + 1$ -tuple of positive integers. Consider the polynomial ring $\mathbb{C}[z_0, \dots, z_k]$ graded by $\deg z_i = n_i$. The scheme $\mathrm{Proj}(\mathbb{C}[z_0, \dots, z_k])$ is said to be the weighted projective space of type N and we denote it by $\mathbb{P}(N)$.

Consider the standard (nonweighted) projective space $\mathbb{P}^k := \mathrm{Proj}(\mathbb{C}[w_0, \dots, w_k])$, where each $\deg w_i = 1$. Then, the graded algebra homomorphism $\mathbb{C}[z_0, \dots, z_k] \rightarrow \mathbb{C}[w_0, \dots, w_k], z_i \mapsto w_i^{n_i}$, induces a morphism $\delta : \mathbb{P}^k \rightarrow \mathbb{P}(N)$.

The following theorem is due to Looijenga [Lo]. His proof had a gap; a complete proof of a more general result is outlined by Bernshtein-Shvartsman [BSh].

3.3 Theorem. *Let \mathcal{C}_1 be an elliptic curve. Then, the variety $(\mathcal{C}_1 \otimes_{\mathbb{Z}} Q^{\vee})/W$ is the weighted projective space of type $(1, a_1^{\vee}, a_2^{\vee}, \dots, a_k^{\vee})$, where a_i^{\vee} are the coefficients of the coroot θ^{\vee} written in terms of the simple coroots $\{\alpha_i^{\vee}\}$ (and, as earlier, k is the rank of G).*

The following table lists the weighted projective space isomorphic to $\mathfrak{M}_{\mathcal{C}_1}(G)$ corresponding to any G . In this table the entries beyond 1 are precisely the numbers $(a_1^{\vee}, a_2^{\vee}, \dots, a_k^{\vee})$ following the convention as in Bourbaki [B, Planche I-IX].

Type of G	Type of the weighted projective space
A_k ($k \geq 1$), C_k ($k \geq 2$)	$(1, 1, 1, \dots, 1)$
B_k ($k \geq 3$)	$(1, 1, 2, \dots, 2, 1)$
D_k ($k \geq 4$)	$(1, 1, 2, \dots, 2, 1, 1)$
G_2	$(1, 1, 2)$
F_4	$(1, 2, 3, 2, 1)$
E_6	$(1, 1, 2, 2, 3, 2, 1)$
E_7	$(1, 2, 2, 3, 4, 3, 2, 1)$
E_8	$(1, 2, 3, 4, 6, 5, 4, 3, 2)$.

We recall the following result from the theory of weighted projective spaces (see, e.g., Beltrametti-Robbiano [BR, Lemma 3B.2.c and Theorem 7.1.c]).

3.4 Theorem. *Let $N = (n_0, \dots, n_k)$ and assume $\gcd\{n_0, \dots, n_k\} = 1$. Then, we have the following.*

(a) $\text{Pic}(\mathbb{P}(N)) \simeq \mathbb{Z}$. In fact, the morphism δ of Definition 3.2 induces an injective map $\delta^* : \text{Pic}(\mathbb{P}(N)) \rightarrow \text{Pic}(\mathbb{P}^k)$.

Moreover, the ample generator of $\text{Pic}(\mathbb{P}(N))$ maps to $\mathcal{O}_{\mathbb{P}^k}(s)$ under δ^* , where s is the least common multiple of $\{n_0, \dots, n_k\}$. We denote this ample generator by $\mathcal{O}_{\mathbb{P}(N)}(s)$.

(b) For any $d \geq 0$,

$$H^0(\mathbb{P}(N), \mathcal{O}_{\mathbb{P}(N)}(s)^{\otimes d}) = \mathbb{C}[z_0, \dots, z_k]_{ds},$$

where $\mathbb{C}[z_0, \dots, z_k]_{ds}$ denotes the subspace of $\mathbb{C}[z_0, \dots, z_k]$ consisting of homogeneous elements of degree ds .

Using Theorems 3.1, 3.3 and 3.4 and the fact that the least common multiple of the numbers $\{1, a_1^\vee, a_2^\vee, \dots, a_k^\vee\}$ for each G is the Dynkin index $m_G = m_{V(\omega_d)}$, we have

$$(*) \quad \Theta_{V(\omega_d)}(\mathcal{C}_1, G) = \mathcal{O}_{\mathbb{P}(1, a_1^\vee, a_2^\vee, \dots, a_k^\vee)}(m_G)^{\otimes p}$$

for some positive integer p . The value of m_G is given in Proposition 2.3 for any G .

We recall the following basic result, the first part of which is due independently to Beauville-Laszlo [BL], Faltings [Fa] and Kumar-Narasimhan-Ramanathan [KNR]. The second part of the theorem as in (1) is the celebrated Verlinde formula for the dimension of the space of conformal blocks essentially due to Tsuchiya-Ueno-Yamada [TUY] (together with works [Fa, Appendix] and [T₁]).

3.5 Theorem. *For any ample line bundle $\mathfrak{L} \in \text{Pic}(\mathfrak{M}_{\mathcal{C}_g}(G))$ and $\ell \geq 0$, there is an isomorphism (canonical up to scalar multiples):*

$$H^0(\mathfrak{M}_{\mathcal{C}_g}(G), \mathfrak{L}^{\otimes \ell}) \simeq L(\mathcal{C}_g, \ell m_{\mathfrak{L}}),$$

where $L(\mathcal{C}_g, \ell)$ is the space of conformal blocks corresponding to the one marked point on \mathcal{C}_g and trivial representation attached to it with central charge ℓ (cf., e.g., [TUY] for the definition of conformal blocks) and $m_{\mathfrak{L}}$ is the positive integer such that $\beta(\mathfrak{L}) = \mathfrak{L}_{\chi_0}^{\otimes m_{\mathfrak{L}}}$, β being the map as in Theorem 2.2.

Moreover, the dimension $F_g(\ell)$ of the space $L(\mathcal{C}_g, \ell)$ is given by the following Verlinde formula:

$$(1) \quad F_g(\ell) = t_\ell^{g-1} \sum_{\mu \in P_\ell} \prod_{\alpha \in \Delta_+} |2 \sin(\frac{\pi}{\ell + h} \langle \alpha, \mu + \rho \rangle)|^{2-2g},$$

where

- $\langle, \rangle :=$ Killing form on \mathfrak{h}^* normalized so that $\langle \theta, \theta \rangle = 2$ for the highest root θ
- $\Delta_+ :=$ the set of positive roots,
- $P_\ell := \{\text{dominant integral weights } \mu \mid \langle \mu, \theta \rangle \leq \ell\}$,
- $\rho :=$ half sum of positive roots,
- $h := \langle \rho, \theta \rangle + 1$, the dual Coxeter number,
- $t_\ell := (\ell + h)^{\text{rank } G} (\#P/Q_{lg})$,

and P is the weight lattice and Q_{lg} is the sublattice of the root lattice Q generated by the long roots.

In fact, we only need to use the above theorem for the case of genus $g = 1$. For $g = 1$, the Verlinde formula (1) clearly reduces to the identity:

$$F_1(\ell) = \#P_\ell.$$

Of course,

$$P_\ell = \{(n_1, \dots, n_k) \in (\mathbb{Z}_+)^k : \sum_{i=1}^k n_i a_i^\vee \leq \ell\}.$$

Proof of Proposition 1.9. Using the specialization of Theorem 3.5 to $g = 1$, we see that

$$\dim H^0(\mathfrak{M}_{\mathcal{C}_1}(G), \Theta_{V(\omega_d)}(\mathcal{C}_1, G)) = \#P_{m_G}.$$

On the other hand, by Theorems 3.1, 3.3 and 3.4(b),

$$\dim H^0(\mathfrak{M}_{\mathcal{C}_1}(G), \mathcal{O}_{\mathbb{P}(1, a_1^\vee, a_2^\vee, \dots, a_k^\vee)}(m_G)^{\otimes p}) = \dim(\mathbb{C}[z_0, \dots, z_k]_{pm_G}) = \#P_{pm_G}.$$

Hence, in the equation (*) following Theorem 3.4, $p = 1$ and $\Theta_{V(\omega_d)}(\mathcal{C}_1, G)$ is the (ample) generator of $\text{Pic}(\mathfrak{M}_{\mathcal{C}_1}(G))$. This proves Proposition 1.9. \square

4 Proof of Proposition 1.7

In this section, we take $G = SL(2)$ and abbreviate $\mathfrak{M}_{\mathcal{C}_g}(SL(2))$ by $\mathfrak{M}_{\mathcal{C}_g}$ etc. Let $\mathfrak{M}_{\mathcal{C}_g}^{\text{red}}$ be the closed subvariety of the moduli space $\mathfrak{M}_{\mathcal{C}_g}$ consisting of decomposable bundles on \mathcal{C}_g (which are semistable of rank-2 with trivial determinant). Let $\mathfrak{J}_{\mathcal{C}_g}$ be the Jacobian of \mathcal{C}_g . Recall that the underlying set of the variety $\mathfrak{J}_{\mathcal{C}_g}$ consists of all the isomorphism classes of line bundles on \mathcal{C}_g with trivial first Chern class. Then, there is a surjective morphism $\xi = \xi_{\mathcal{C}_g} : \mathfrak{J}_{\mathcal{C}_g} \rightarrow \mathfrak{M}_{\mathcal{C}_g}^{\text{red}} \subset \mathfrak{M}_{\mathcal{C}_g}$, taking $\mathfrak{L} \mapsto \mathfrak{L} \oplus \mathfrak{L}^{-1}$. Moreover, $\xi^{-1}(\xi(\mathfrak{L})) = \{\mathfrak{L}, \mathfrak{L}^{-1}\}$. The Jacobian $\mathfrak{J}_{\mathcal{C}_g}$ admits the involution τ taking $\mathfrak{L} \mapsto \mathfrak{L}^{-1}$.

Let T be a maximal torus of the maximal compact subgroup $SU(2)$ of $SL(2)$, which we take to be the diagonal subgroup of $SU(2)$. Similar to the identification $\theta_{\mathcal{C}_g}$ as in Theorem 1.5, setting $J_g := T^{2g}$, there is an isomorphism of real analytic spaces $\bar{\theta}_{\mathcal{C}_g} : J_g \rightarrow \mathfrak{J}_{\mathcal{C}_g}$ making the following diagram commutative:

$$\begin{array}{ccc} & J_g \xrightarrow{\bar{\theta}_{\mathcal{C}_g}} \mathfrak{J}_{\mathcal{C}_g} & \\ (E) & f_g \downarrow & \downarrow \xi_{\mathcal{C}_g} \\ & M_g \xrightarrow{\theta_{\mathcal{C}_g}} \mathfrak{M}_{\mathcal{C}_g}, & \end{array}$$

where $f_g : J_g \rightarrow M_g$ is induced from the standard inclusion $T^{2g} \subset SU(2)^{2g}$. We will explicitly describe the isomorphism $\bar{\theta}_{\mathcal{C}_g}$ in the proof of the following lemma.

Recall the definition of the map $i_g : M_g \rightarrow M_{g+1}$ from Section 1 and let $r_g : J_g \rightarrow J_{g+1}$ be the map $(t_1, \dots, t_{2g}) \mapsto (t_1, \dots, t_{2g}, 1, 1)$. Then, we have the following commutative diagram:

$$(F) \quad \begin{array}{ccc} J_g & \xrightarrow{f_g} & M_g \\ r_g \downarrow & & \downarrow i_g \\ J_{g+1} & \xrightarrow{f_{g+1}} & M_{g+1}. \end{array}$$

Let x_{g+1} denote the positive generator of $H^2(M_{g+1}, \mathbb{Z})$. Then, by Lemma 1.4 and Theorem 1.5,

$$i_g^*(x_{g+1}) = d_g x_g,$$

for some integer d_g . We will prove that $d_g = 1$, which will of course prove Proposition 1.7. Set $y_g := f_g^*(x_g)$; $f_g^* : H^2(M_g, \mathbb{Z}) \rightarrow H^2(J_g, \mathbb{Z})$ being the map in cohomology induced from f_g .

4.1 Lemma. $y_g \neq 0$ and $r_g^*(y_{g+1}) = y_g$ as elements of $H^2(J_g, \mathbb{Z})$.

Proof. There exists a unique universal line bundle \mathcal{P} , called the *Poincaré bundle* on $\mathcal{C}_g \times \mathfrak{J}_{\mathcal{C}_g}$ such that, for each $\mathfrak{L} \in \mathfrak{J}_{\mathcal{C}_g}$, \mathcal{P} restricts to the line bundle \mathfrak{L} on $\mathcal{C}_g \times \mathfrak{L}$, and \mathcal{P} restricted to $x_o \times \mathfrak{J}_{\mathcal{C}_g}$ is trivial for a fixed base point $x_o \in \mathcal{C}_g$ (cf. [ACGH, Chap. IV, §2]).

Let \mathcal{F} be the rank-2 vector bundle $\mathcal{P} \oplus \hat{\tau}^*(\mathcal{P})$ over the base space $\mathcal{C}_g \times \mathfrak{J}_{\mathcal{C}_g}$, and think of \mathcal{F} as a family of rank-2 bundles on \mathcal{C}_g parametrized by $\mathfrak{J}_{\mathcal{C}_g}$, where $\hat{\tau} : \mathcal{C}_g \times \mathfrak{J}_{\mathcal{C}_g} \rightarrow \mathcal{C}_g \times \mathfrak{J}_{\mathcal{C}_g}$ is the involution $I \times \tau$.

By Drezet-Narasimhan [DN], we have $x_g = c_1(\Theta_{V_2}(\mathcal{C}_g, SL(2)))$ for the standard representation V_2 of $SL(2)$. Using the functoriality of Chern class,

$$(1) \quad \xi_{\mathcal{C}_g}^*(x_g) = c_1(\text{Det } \mathcal{F}),$$

where $\text{Det } \mathcal{F}$ denotes the determinant line bundle over $\mathfrak{J}_{\mathcal{C}_g}$ associated to the family \mathcal{F} (cf. Definition 1.2). Recall that the fiber of $\text{Det } \mathcal{F}$ at any $\mathfrak{L} \in \mathfrak{J}_{\mathcal{C}_g}$ is given by the expression

$$(2) \quad \begin{aligned} \text{Det } \mathcal{F}|_{\mathfrak{L}} &= \wedge^{\text{top}} (H^0(\mathcal{C}_g, \mathfrak{L} \oplus \mathfrak{L}^{-1})^*) \otimes \wedge^{\text{top}} (H^1(\mathcal{C}_g, \mathfrak{L} \oplus \mathfrak{L}^{-1})) \\ &= \wedge^{\text{top}} (H^0(\mathcal{C}_g, \mathfrak{L})^* \oplus H^0(\mathcal{C}_g, \mathfrak{L}^{-1})^*) \otimes \wedge^{\text{top}} (H^1(\mathcal{C}_g, \mathfrak{L}) \oplus H^1(\mathcal{C}_g, \mathfrak{L}^{-1})) \\ &= \wedge^{\text{top}} (H^0(\mathcal{C}_g, \mathfrak{L})^*) \otimes \wedge^{\text{top}} (H^0(\mathcal{C}_g, \mathfrak{L}^{-1})^*) \otimes \wedge^{\text{top}} (H^1(\mathcal{C}_g, \mathfrak{L})) \otimes \wedge^{\text{top}} (H^1(\mathcal{C}_g, \mathfrak{L}^{-1})) \\ &= (\text{Det } \mathcal{P})|_{\mathfrak{L}} \otimes (\tau^*(\text{Det } \mathcal{P}))|_{\mathfrak{L}}. \end{aligned}$$

Applying the Grothendieck-Riemann-Roch theorem (cf. [F, Example 15.2.8]) for the projection $\mathcal{C}_g \times \mathfrak{J}_{\mathcal{C}_g} \xrightarrow{\pi} \mathfrak{J}_{\mathcal{C}_g}$ gives

$$(3) \quad \text{ch}(R\pi_*\mathcal{P}) = \pi_*(\text{ch } \mathcal{P} \cdot \text{Td } T_\pi),$$

where ch is the Chern character and $\text{Td } T_\pi$ denotes the Todd genus of the relative tangent bundle of $\mathcal{C}_g \times \mathfrak{J}_{\mathcal{C}_g}$ along the fibers of π . By the definition of $\text{Det } \mathcal{P}$ and $R\pi_*\mathcal{P}$,

$$(4) \quad c_1(\text{Det } \mathcal{P}) = -\text{ch}(R\pi_*\mathcal{P})_{[2]},$$

where, for a cohomology class y , $y_{[n]}$ denotes the component of y in H^n . Since \mathcal{P} restricted to $x_o \times \mathfrak{J}_{\mathcal{C}_g}$ is trivial and for any $\mathfrak{L} \in \mathfrak{J}_{\mathcal{C}_g}$, \mathcal{P} restricts to the line bundle \mathfrak{L} on $\mathcal{C}_g \times \mathfrak{L}$ (with the trivial Chern class), we get

$$(5) \quad c_1(\mathcal{P}) \in H^1(\mathcal{C}_g) \otimes H^1(\mathfrak{J}_{\mathcal{C}_g}).$$

Thus, using (3)-(4),

$$(6) \quad \begin{aligned} -c_1(\text{Det } \mathcal{P}) &= \pi_*((\text{ch } \mathcal{P} \cdot \text{Td } T_\pi)_{[4]}) \\ &= \pi_*\left(\frac{c_1(\mathcal{P})^2}{2} + \frac{c_1(\mathcal{P}) \cdot c_1(T_\pi)}{2}\right) \\ &= \pi_*(c_1(\mathcal{P})^2) / 2. \end{aligned}$$

The last equality follows from (5), since the cup product $c_1(\mathcal{P}) \cdot c_1(T_\pi)$ vanishes, $c_1(T_\pi)$ being in $H^2(\mathcal{C}_g) \otimes H^0(\mathfrak{J}_{\mathcal{C}_g})$.

Recall the presentation of $\pi_1(\mathcal{C}_g)$ given just above Theorem 1.5. Then, $H_1(\mathcal{C}_g, \mathbb{Z}) = \oplus_{i=1}^g \mathbb{Z}a_i \oplus \oplus_{i=1}^g \mathbb{Z}b_i$. Moreover, the \mathbb{Z} -module dual basis $\{a_i^*, b_i^*\}_{i=1}^g$ of $H^1(\mathcal{C}_g, \mathbb{Z}) = \text{Hom}_{\mathbb{Z}}(H_1(\mathcal{C}_g, \mathbb{Z}), \mathbb{Z})$ satisfies $a_i^* \cdot a_j^* = 0 = b_i^* \cdot b_j^*$, $a_i^* \cdot b_j^* = \delta_{ij}[\mathcal{C}_g]$, where $[\mathcal{C}_g]$ denotes the positive generator of $H^2(\mathcal{C}_g, \mathbb{Z})$.

Having fixed a base point x_o in \mathcal{C}_g , define the algebraic map

$$\psi : \mathcal{C}_g \rightarrow \mathfrak{J}_{\mathcal{C}_g}, \quad x \mapsto \mathcal{O}(x - x_o).$$

Of course, $\mathfrak{J}_{\mathcal{C}_g}$ is canonically identified as $H^1(\mathcal{C}_g, \mathcal{O}_{\mathcal{C}_g})/H^1(\mathcal{C}_g, \mathbb{Z})$. Thus, as a real analytic space, we can identify

$$(7) \quad \mathfrak{J}_{\mathcal{C}_g} \simeq H^1(\mathcal{C}_g, \mathbb{R})/H^1(\mathcal{C}_g, \mathbb{Z}) \simeq H^1(\mathcal{C}_g, \mathbb{Z}) \otimes_{\mathbb{Z}} (\mathbb{R}/\mathbb{Z}) \simeq \text{Hom}_{\mathbb{Z}}(H_1(\mathcal{C}_g, \mathbb{Z}), \mathbb{R}/\mathbb{Z}) = J_g$$

obtained from the \mathbb{R} -vector space isomorphism

$$H^1(\mathcal{C}_g, \mathbb{R}) \simeq H^1(\mathcal{C}_g, \mathcal{O}_{\mathcal{C}_g}),$$

induced from the inclusion $\mathbb{R} \subset \mathcal{O}_{\mathcal{C}_g}$, where the last equality in (7) follows by using the basis $\{a_1, b_1, \dots, a_g, b_g\}$ of $H_1(\mathcal{C}_g, \mathbb{Z})$. The induced map, under the identification (7),

$$\psi_* : H_1(\mathcal{C}_g, \mathbb{Z}) \rightarrow H_1(\mathfrak{J}_{\mathcal{C}_g}, \mathbb{Z}) \simeq H^1(\mathcal{C}_g, \mathbb{Z})$$

is the Poincaré duality isomorphism. To see this, identify

$$(8) \quad \text{Hom}_{\mathbb{Z}}(H_1(\mathcal{C}_g, \mathbb{Z}), \mathbb{R}/\mathbb{Z}) \simeq \text{Hom}_{\mathbb{Z}}(H^1(\mathcal{C}_g, \mathbb{Z}), \mathbb{R}/\mathbb{Z})$$

using the Poincaré duality isomorphism: $H_1(\mathcal{C}_g, \mathbb{Z}) \simeq H^1(\mathcal{C}_g, \mathbb{Z})$. Then, under the identifications (7)-(8), the map

$$\psi : \mathcal{C}_g \rightarrow \text{Hom}_{\mathbb{Z}}(H^1(\mathcal{C}_g, \mathbb{Z}), \mathbb{R}/\mathbb{Z})$$

can be described as

$$\psi(x)([\omega]) = e^{2\pi i \int_{x_o}^x \omega},$$

for any closed 1-form ω on \mathcal{C}_g representing the cohomology class $[\omega] \in H^1(\mathcal{C}_g, \mathbb{Z})$ (cf. [M, Theorem 2.5]), where $\int_{x_o}^x \omega$ denotes the integral of ω along any path in \mathcal{C}_g from x_o to x .

Since

$$\psi_* : H_1(\mathcal{C}_g, \mathbb{Z}) \rightarrow H_1(\mathfrak{J}_{\mathcal{C}_g}, \mathbb{Z}) \simeq H^1(\mathcal{C}_g, \mathbb{Z})$$

is the Poincaré duality isomorphism, it is easy to see that the induced cohomology map

$$\psi^* : H^1(\mathfrak{J}_{\mathcal{C}_g}, \mathbb{Z}) \simeq H_1(\mathcal{C}_g, \mathbb{Z}) \rightarrow H^1(\mathcal{C}_g, \mathbb{Z})$$

is given by

$$(9) \quad \psi^*(a_i) = -b_i^*, \quad \psi^*(b_i) = a_i^* \quad \text{for all } 1 \leq i \leq g.$$

In particular, ψ^* is an isomorphism. Moreover, the isomorphism does not depend on the choice of x_o .

Consider the map

$$\mathcal{C}_g \times \mathcal{C}_g \xrightarrow{I \times \psi} \mathcal{C}_g \times \mathfrak{J}_{\mathcal{C}_g}.$$

Let $\mathcal{P}' := (I \times \psi)^*(\mathcal{P})$. Then, \mathcal{P}' is the unique line bundle over $\mathcal{C}_g \times \mathcal{C}_g$ satisfying the following properties:

$$\mathcal{P}'|_{\mathcal{C}_g \times x} = \mathcal{O}(x - x_o) \quad \text{and} \quad \mathcal{P}'|_{x_o \times \mathcal{C}_g} \text{ is trivial.}$$

Consider the following line bundle over $\mathcal{C}_g \times \mathcal{C}_g$:

$$\mathcal{O}_{\mathcal{C}_g \times \mathcal{C}_g}(\Delta) \otimes (\mathcal{O}(-x_o) \boxtimes 1) \otimes (1 \boxtimes \mathcal{O}(-x_o)),$$

where Δ denotes the diagonal in $\mathcal{C}_g \times \mathcal{C}_g$. One sees that this bundle also satisfies the restriction properties mentioned above and hence it must be isomorphic with \mathcal{P}' . Consequently,

$$c_1(\mathcal{P}') = c_1(\mathcal{O}_{\mathcal{C}_g \times \mathcal{C}_g}(\Delta)) + c_1(\mathcal{O}(-x_o) \boxtimes 1) + c_1(1 \boxtimes \mathcal{O}(-x_o)).$$

Using the definition of \mathcal{P}' and the functoriality of the Chern classes,

$$(10) \quad c_1(\mathcal{P}') = c_1((I \times \psi)^*(\mathcal{P})) = (I \times \psi)^*c_1(\mathcal{P}).$$

By (5), $c_1(\mathcal{P}) \in H^1(\mathcal{C}_g) \otimes H^1(\mathfrak{J}_{\mathcal{C}_g})$, and hence $c_1(\mathcal{P}') \in H^1(\mathcal{C}_g) \otimes H^1(\mathcal{C}_g)$. Moreover,

$$c_1(\mathcal{O}(-x_o) \boxtimes 1) + c_1(1 \boxtimes \mathcal{O}(-x_o)) \in H^2(\mathcal{C}_g) \otimes H^0(\mathcal{C}_g) \oplus H^0(\mathcal{C}_g) \otimes H^2(\mathcal{C}_g).$$

Thus, $c_1(\mathcal{P}')$ is the component of $c_1(\mathcal{O}_{\mathcal{C}_g \times \mathcal{C}_g}(\Delta))$ in $H^1(\mathcal{C}_g) \otimes H^1(\mathcal{C}_g)$. Hence, by Milnor-Stasheff [MS, Theorem 11.11],

$$c_1(\mathcal{P}') = - \sum_{i=1}^g a_i^* \otimes b_i^* + \sum_{i=1}^g b_i^* \otimes a_i^*.$$

Therefore, by (10),

$$c_1(\mathcal{P}) = - \sum_{i=1}^g a_i^* \otimes \psi^{*-1}(b_i^*) + \sum_{i=1}^g b_i^* \otimes \psi^{*-1}(a_i^*),$$

and thus, by (6),

$$\begin{aligned} c_1(\text{Det } \mathcal{P}) &= -\frac{1}{2}\pi_*(c_1(\mathcal{P})^2) \\ &= -\frac{1}{2}\pi_*\left(\left(-\sum_{i=1}^g a_i^* \otimes \psi^{*-1}(b_i^*) + \sum_{i=1}^g b_i^* \otimes \psi^{*-1}(a_i^*)\right)^2\right) \\ &= -\frac{1}{2}\pi_*\left(\sum_{i=1}^g a_i^* \cdot b_i^* \otimes \psi^{*-1}(b_i^*) \cdot \psi^{*-1}(a_i^*) + \sum_{i=1}^g b_i^* \cdot a_i^* \otimes \psi^{*-1}(a_i^*) \cdot \psi^{*-1}(b_i^*)\right) \\ &= -\sum_{i=1}^g \psi^{*-1}(b_i^*) \cdot \psi^{*-1}(a_i^*) \in H^2(\mathfrak{J}_{\mathcal{C}_g}, \mathbb{Z}). \end{aligned}$$

Now, the involution τ of $\mathfrak{J}_{\mathcal{C}_g}$ induces the map $-I$ on $H^1(\mathfrak{J}_{\mathcal{C}_g}, \mathbb{Z})$ (since, under the identification $\bar{\theta}_{\mathcal{C}_g} : J_g \rightarrow \mathfrak{J}_{\mathcal{C}_g}$, τ corresponds to the map $x \mapsto x^{-1}$ for $x \in J_g$). Therefore,

$$\tau^*(c_1(\text{Det } \mathcal{P})) = c_1(\text{Det } \mathcal{P}).$$

Hence, by the identities (1)-(2),

$$\begin{aligned} \xi_{\mathcal{C}_g}^*(x_g) &= c_1(\text{Det } \mathcal{F}) \\ &= 2c_1(\text{Det } \mathcal{P}) \\ (11) \quad &= 2 \sum_{i=1}^g \psi^{*-1}(a_i^*) \cdot \psi^{*-1}(b_i^*), \end{aligned}$$

which is clearly a nonvanishing class in $H^2(\mathfrak{J}_{\mathcal{C}_g}, \mathbb{Z})$. Moreover, for any $g \geq 2$, under the identification (7), the map $r_{g-1} : J_{g-1} \rightarrow J_g$ corresponds to the map $H_1(\mathcal{C}_g, \mathbb{Z}) \rightarrow H_1(\mathcal{C}_{g-1}, \mathbb{Z})$, $a_i \mapsto a_i, b_i \mapsto b_i$ for $1 \leq i \leq g-1$, $a_g \mapsto 0, b_g \mapsto 0$. Thus, by (9) and (11), $\xi_{\mathcal{C}_g}^*(x_g)$ restricts, via r_{g-1}^* , to the class $\xi_{\mathcal{C}_{g-1}}^*(x_{g-1})$ for any $g \geq 2$. But, by the commutative diagram (E), $\xi_{\mathcal{C}_g}^*(x_g) = y_g$. This proves Lemma 4.1. \square

Proof of Proposition 1.7. By the above Lemma 4.1 and the commutative diagram (F), we see that

$$f_g^*(d_g x_g) = f_g^* \iota_g^*(x_{g+1}) = r_g^*(f_{g+1}^*(x_{g+1})), \text{ i.e., } d_g y_g = y_g.$$

Since the cohomology of J_g is torsion free and y_g is a nonvanishing class, we get $d_g = 1$. This concludes the proof of Proposition 1.7. \square

References

- [A] Adams, F., *Lectures on Lie Groups*, W.A. Benjamin, Inc., 1969.
- [ACGH] Arbarello, E., Cornalba, M., Griffiths, P. and Harris, J., *Geometry of Algebraic Curves*, Springer-Verlag, 1985.
- [B] Bourbaki, N., *Groupes et Algèbres de Lie*, Chap. 4–6, Masson, Paris, 1981.
- [BL] Beauville, A. and Laszlo, Y., *Conformal blocks and generalized theta functions*, Commun. Math. Phys. **164**, 385–419, (1994).
- [BLS] Beauville, A., Laszlo, Y. and Sorger, C., *The Picard group of the moduli of G -bundles on a curve*, Compositio Math. **112**, 183–216, (1998).
- [BR] Beltrametti, M. and Robbiano, L., *Introduction to the theory of weighted projective spaces*, Expo. Math. **4**, 111–162, (1986).
- [BSh] Bernshtein, I.N. and Shvartsman, O.V., *Chevalley’s Theorem for complex crystallographic Coxeter groups*, Funct. Anal. Appl. **12**, 308–310, (1978).
- [DN] Drezet, J.–M. and Narasimhan, M.S., *Groupe de Picard des variétés de modules de fibrés semi-stables sur les courbes algébriques*, Invent. Math. **97**, 53–94, (1989).
- [D] Dynkin, E.B., *Semisimple subalgebras of semisimple Lie algebras*, Am. Math. Soc. Transl. (Ser. II) **6**, 111–244, (1957).
- [Fa] Faltings, G., *A proof for the Verlinde formula*, J. Alg. Geom. **3**, 347–374, (1994).
- [F] Fulton, W., *Intersection Theory*, Springer, 1998.
- [FMW] Friedman, R., Morgan, J.W. and Witten, E., *Principal G -bundles over elliptic curves*, Mathematical Research Letters **5**, 97–118, (1998).
- [KM] Knudsen, F. and Mumford, D., *The projectivity of the moduli space of stable curves I: preliminaries on “det” and “div”*, Math. Scand. **39**, 19–55, (1976).
- [K₁] Kumar, S., *Infinite Grassmannians and moduli spaces of G -bundles*, in: *Vector Bundles on Curves - New Directions*, Lecture Notes in Mathematics **1649**, Springer-Verlag, 1–49, (1997).
- [K₂] Kumar, S., *Kac-Moody Groups, their Flag Varieties and Representation Theory*, Progress in Mathematics vol. **204**, Birkhauser, 2002.
- [KN] Kumar, S. and Narasimhan, M.S., *Picard group of the moduli spaces of G -bundles*, Math. Annalen **308**, 155–173, (1997).
- [KNR] Kumar, S., Narasimhan, M.S., and Ramanathan, A., *Infinite Grassmannians and moduli spaces of G -bundles*, Math. Annalen **300**, 41–75, (1994).
- [L] Lang, S., *Introduction to Arakelov Theory*, Springer, 1988.
- [La] Laszlo, Y., *About G -bundles over elliptic curves*, Ann. Inst. Fourier, Grenoble **48**, 413–424, (1998).

- [Lo] Looijenga, E., *Root systems and elliptic curves*, *Invent. Math.* **38**, 17–32, (1976).
- [LS] Laszlo, Y. and Sorger, C., *The line bundles on the moduli of parabolic G -bundles over curves and their sections*, *Ann. Scient. Éc. Norm. Sup.* **30**, 499–525, (1997).
- [M] Milne, J.S., Chap. VII – *Jacobian varieties*, in: *Arithmetic Geometry* (Cornell, G., et al., eds.), Springer-Verlag, 167–212, (1986).
- [MS] Milnor, J.W. and Stasheff, J.D., *Characteristic Classes*, *Annals of Mathematics Studies* vol. 76, Princeton University Press, 1974.
- [N] Narasimhan, R., *Compact Riemann Surfaces*, Birkhauser Verlag, 1992.
- [NS] Narasimhan, M.S. and Seshadri, C.S., *Stable and unitary vector bundles on a compact Riemann surface*, *Ann. of Math.* **82**, 540–567, (1965).
- [RR] Ramanan, S. and Ramanathan, A., *Some remarks on the instability flag*, *Tôhoku Math. J.* **36**, 269–291, (1984).
- [R₁] Ramanathan, A., *Stable principal bundles on a compact Riemann surface-Construction of moduli space*, Thesis, University of Bombay, 1976.
- [R₂] Ramanathan, A., *Stable principal bundles on a compact Riemann surface*, *Math. Annalen* **213**, 129–152, (1975).
- [So] Sorger, C., *On moduli of G -bundles on a curve for exceptional G* , *Ann. Scient. Éc. Norm. Sup.* **32**, 127–133, (1999).
- [S] Szenes, A., *The combinatorics of the Verlinde formula*, in: *Vector Bundles in Algebraic Geometry* (Hitchin, N.J., et al., eds.), CUP **241**, (1995).
- [T₁] Teleman, C., *Lie algebra cohomology and the fusion rules*, *Commun. Math. Phys.* **173**, 265–311, (1995).
- [T₂] Teleman, C., *Borel-Weil-Bott theory on the moduli stack of G -bundles over a curve*, *Invent. Math.* **134**, 1–57, (1998).
- [TUY] Tsuchiya, A., Ueno, K. and Yamada, Y., *Conformal field theory on universal family of stable curves with gauge symmetries*, *Adv. Stud. Pure Math.* **19**, 459–565, (1989).

Address (of both the Authors): Department of Mathematics, University of North Carolina, Chapel Hill, NC 27599-3250, USA
 shrawan@email.unc.edu
 boysal@email.unc.edu